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Manuscript received August 9, 1943; revision received September 25 and accepted September 26, 1973.

A Simple Algorithm for the Time-Optimal Control of Chemical Processes

A simple algorithm for the time-optimal control of chemical processes during setpoint changes, in processes which can be described by a second-order lag plus dead time model, is described. Knowledge of the unsteady state model parameters is not required because the algorithm uses a dimensionless phase plane on which the switching curves are independent of system parameters for a given forcing function. The algorithm gives the parameters of a second-order lag plus dead time model as a byproduct of the setpoint change. It is easily tuned and is relatively insensitive to changes in the process dynamics. The algorithm does not require a large computer or long computing times and has been implemented on both analog and digital computers in controlling computer simulated systems.

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SCOPE

The time-optimal control problem is basically one of determining the control action that will drive a process from an initial state to a specified final state in minimum time. A time-optimal setpoint change for a second-order system is accomplished by applying full forward forcing at time $t = 0$, switching to full reverse forcing at time t_1, and returning to conventional setpoint control at t_2. The problem is to determine the times at which to make the switches, t_1 and t_2.

Time-optimal control offers economic advantages for many types of chemical processes. At plants operating under supervisory computer control, the computer often makes setpoint changes. When the supervisory computer specifies more profitable operating conditions, it is desirable to bring the plant to those new conditions as quickly as possible. This is particularly important if the time between the plant disturbances which cause a change in the optimum operating conditions is comparable to the time required to line the plant out at a new set of operating conditions. In the operation of batch processes, step changes in process variables are often required. Undue delay in completing these changes decreases the capacity of the unit and should be avoided.

In attempting to apply time-optimal control to a process, two obstacles are encountered: (1) An unsteady state mathematical model of the system is required and (2) a nonlinear, multipoint, boundary-value problem, with unspecified final time must usually be solved in order to determine the switching times.

Most of the existing methods of obtaining the time-optimal switching times avoid the boundary-value problem but require prior knowledge of unsteady state model parameters. The objective of this study was to develop an algorithm that does not require the knowledge of these parameters.

CONCLUSIONS AND SIGNIFICANCE

The results obtained show that it is possible to generalize the solution of time-optimal control problem into an algorithm that does not require knowledge of unsteady state model parameters. In the traditional phase plane representation of the time-optimal control problem, the switching curves that determine t_1 are functions of the

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system parameters; therefore, each system requires a different switching curve. The modeling time-optimal controller algorithm is based on the use of a dimensionless phase plane in which $t\ dX/dt$, instead of dX/dt, is plotted versus X (the fraction of the setpoint change completed). In this dimensionless phase plane a switching curve can be obtained which is independent of system parameters for a given forcing function. Since model parameters are subject to change due to nonlinearities or changes in the process, the fact that model parameters are not needed gives this algorithm a distinct advantage over other methods. A steady state model of the system is required in order to predict steady state values of the controller output. The higher-order systems are controlled as if they were second-order systems with dead time. In addition, the modeling, time-optimal controller provides the parameters of a second-order lag plus dead time model of the system as a byproduct of a time-optimal setpoint change. This model closely approximates the step responses, as well as the trajectories and switching times of the process during time-optimal control. The modeling, time-optimal controller is easily tuned and is relatively insensitive to changes in the process dynamics. Even though the dynamics of the system may change, the algorithm compensates for much of the change and does not require retuning. This is because the tuning itself is a correction for deviation from second-order behavior.

Implementation of the algorithm using analog components would allow time-optimal control of processes not having a digital computer available.

This algorithm has been demonstrated on both analog and digital computers in controlling and modeling linear second-order overdamped and underdamped systems; linear third- and fourth-order systems with dead time; and a highly nonlinear exothermic continuous stirred-tank reactor with dead time. All demonstrations were of computer simulated systems.

Time domain correlations, that can be used to implement the open-loop time-optimal control of systems where the parameters of a second-order plus dead time model describing the system are known, were developed.

The time-optimal or minimum time control problem is basically one of determining the control action that will drive a process from an initial state to a specified final state in minimum time. Bellman et al. (1956) have shown that the optimum control action will be on-off or bang-bang control. The general shape of a time-optimal response to a setpoint change is shown in Figure 1 where it is compared with a simple step response and with a response using a proportional-integral-derivative (PID) mode controller tuned to minimize the integral of the absolute error. The control actions which produced these responses are shown in Figure 2. It should be noted that the time-optimal response reaches the new steady state quickly and without any overshoot. A time-optimal control setpoint change for a second-order system is accomplished by applying full forward forcing at time $t = 0$, switching to full reverse forcing at time t_1, and returning to conventional setpoint control at t_2. The problem is to determine the times at which to make the switches, t_1 and t_2. Time-optimal control can be open-loop or programmed which uses precomputed switching times or it can be closed-loop or feedback which uses measured state variables to compute the switching times after the setpoint change has started.

In attempting to apply time-optimal control to a process, two obstacles are encountered:

1. An unsteady state mathematical model of the system is required, and

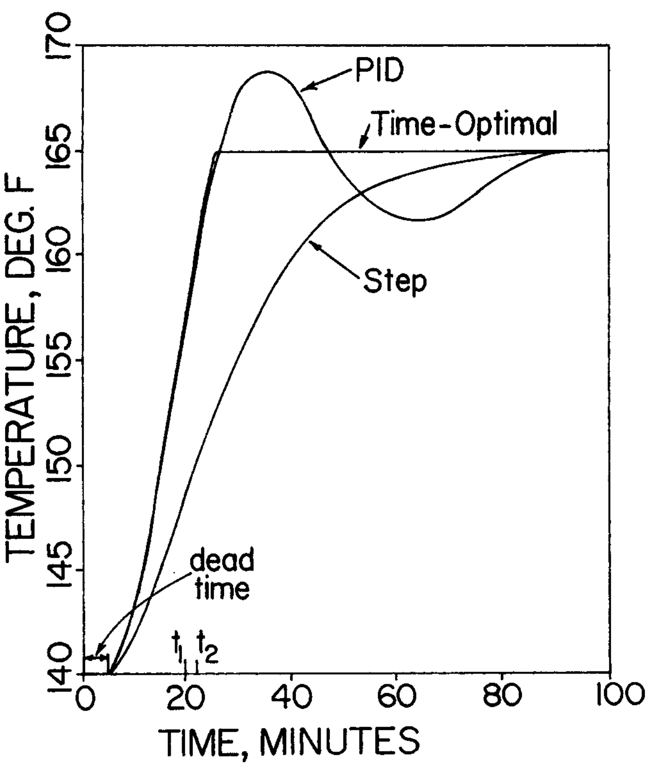

Fig. 1. System setpoint change responses produced by the control actions shown in Figure 2.

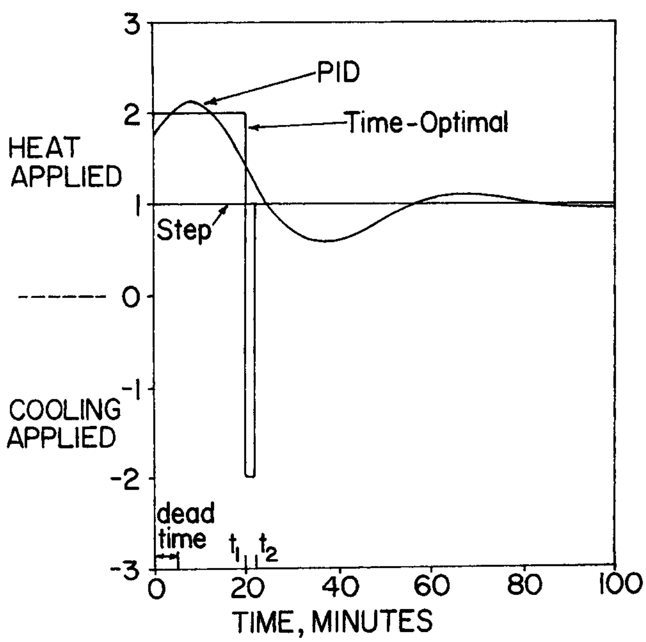

Fig. 2. Control actions used to produce the system responses shown in Figure 1.

2. A nonlinear, multi-point, boundary-value problem with unspecified final time must usually be solved in order to determine the switching times.

Most of the existing methods of obtaining the time-optimal switching times require prior knowledge of unsteady state model parameters. Koppel and Latour (1965) have shown that for a second-order lag plus dead time process model, the equations can be solved analytically and the multi-point, boundary-value problem can be avoided.

Latour et al. (1968) and Douglas and Denn (1965) describe feedback time-optimal controllers which use a completely predetermined unsteady state model of the system. Javinsky and Kadlec (1970) compared the time-optimal control of an experimental laboratory sized reactor with the time-optimal control of an analog computer simulation of the reactor.

In a recent paper Steadman and Koppel (1972) describe a method for obtaining switching times using model parameters determined by curve fitting during a previous setpoint change. Hsu et al. (1972) describe a similar method which determines the model parameters by curve fitting the initial portion of the trajectory after the time-optimal step change has started.

This paper describes a simple, modeling, time-optimal control algorithm which uses state variable feedback to carry out the time-optimal control of second-order systems and suboptimal* control of higher-order systems which can be approximated by second-order lag plus dead time models, without determining unsteady-state parameters. Many chemical operations including cracking furnaces (Lapse, 1956), large distillation columns (Lupfer and Parsons, 1962), heat exchangers (Hougen, 1964), and liquid-liquid extraction columns (Biery and Boylan, 1963), may be approximated by this model.

DESCRIPTION OF THE MODELING TIME-OPTIMAL CONTROLLER ALGORITHM

The differential equation describing the time-optimal control of a linear second-order system is

$$\frac{d^2X}{dt^2} + B_1 \frac{dX}{dt} + B_2 X = B_2 M \quad (1)$$

$$\text{at} \quad t = 0, \quad X = 0, \quad \frac{dX}{dt} = 0$$

$$\text{at} \quad t = t_2, \quad X = 1, \quad \frac{dX}{dt} = 0$$

where $M = M_1$ during the interval $0 \leq t \leq t_1$ and $M = M_2$ during the interval $t_1 \leq t \leq t_2$. Here B_1, B_2, M_1, and M_2 are constants and X is the fraction of the setpoint change completed. In this paper, unless otherwise stated, the ratio $M_2/M_1 = -1$ is used; therefore, time-optimal control with $M = 2$ implies that the forcing function is $2B_2$ prior to t_1 and $-2B_2$ during the interval between t_1 and t_2. This is only for convenience in discussion and does not limit the algorithm. These equations can be put into dimensionless form

$$\frac{d^2X}{d\bar{t}^2} + \frac{dX}{d\bar{t}} + AX = AM \quad (2)$$

where

$$\bar{t} = B_1 t \quad \text{and} \quad A = \frac{B_2}{B_1^2}$$

In the traditional phase plane representation of the time-optimal control problem (Koppel and Latour, 1965), the switching curves that determine t_1, the time to switch from M_1 to M_2, are functions of B_1, B_2, M_1 and M_2; therefore each system requires a different switching curve. The modeling time-optimal controller algorithm is based on the use of a dimensionless phase plane in which $t\ dX/dt$, instead of dX/dt, is plotted versus X. In this dimensionless phase plane a switching curve can be obtained which is independent of the system parameters A, B_1, and B_2 for fixed values of M and the ratio M_2/M_1. It should be noted that M and the ratio M_2/M_1 are under control of the operator and may therefore be manipulated at will. In the range of forcing functions that is of practical interest the switching curves are not very sensitive to the value of M and in some cases may be represented by a single average curve for a given ratio M_2/M_1.

A switching curve for $M = 2$ and $M_2/M_1 = -1$ and the time-optimal trajectories of two different systems, Equation (3):

$$\frac{d^2X}{dt^2} + 0.1667 \frac{dX}{dt} + 0.00667\, X = 0.00667\, M \quad (3)$$

or in the Laplace domain

$$G_{(s)} = \frac{1}{(15s+1)(10s+1)}$$

and Equation (4)

$$\frac{d^2X}{dt^2} + 1.2 \frac{dX}{dt} + 0.2\, X = 0.2\, M \quad (4)$$

or in the Laplace domain

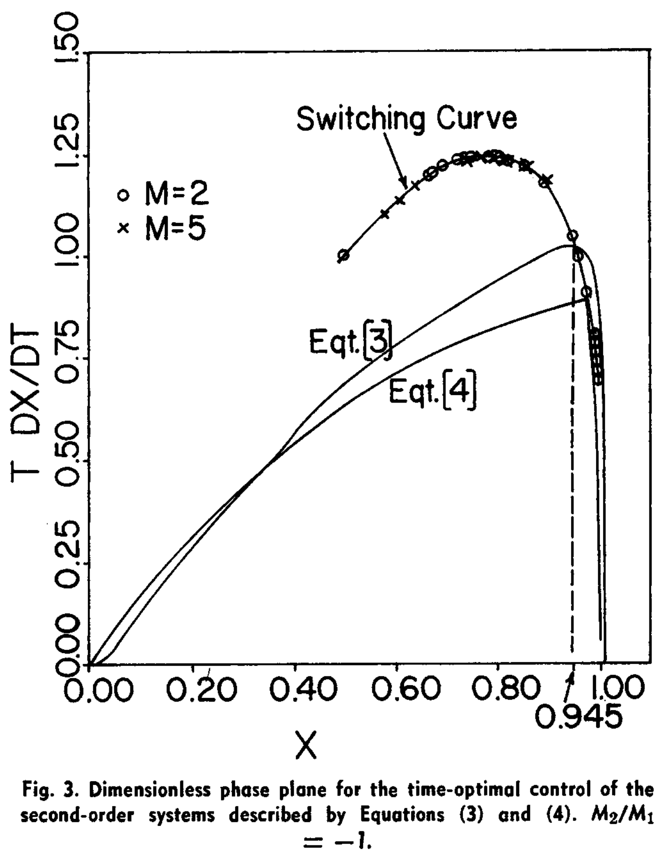

Fig. 3. Dimensionless phase plane for the time-optimal control of the second-order systems described by Equations (3) and (4). $M_2/M_1 = -1$.

* Suboptimal control, as used here, means that the controller will switch only once between the extremes of the manipulated variable. True time-optimal control for linear systems with real eigenvalues requires n-1 switches between extremes, where n is the order of the system. Throughout this paper the term time-optimal is used in a generic sense to include true time-optimal, suboptimal, and other bang-bang controls which approximate true time-optimal control.

$$G_{(s)} = \frac{1}{(5s+1)(s+1)}$$

are shown in Figure 3. Points on the $M = 5$, $M_2/M_1 = -1$ switching curve are also shown. These switching curves were obtained by solving the time-optimal control problem for various values of A and M for $M_2/M_1 = -1$. The values of t and X for the process can be measured directly and dX/dt can be easily approximated since it is nearly constant during most of the trajectory prior to t_1. The controller simply compares $t\ dX/dt$ with the switching curve $f_{(X)}$ and switches between the extremes of the manipulated variable when $t\ dX/dt \geqq f_{(X)}$. Data for similar curves at other ratios of M_2/M_1 may be obtained by solving the time-optimal control problem using the techniques described by Koppel and Latour (1965). Switching curves for $M_1 = 2$, $M_2 = 0$ and $M_1 = 5$, $M_2 = 0$ are shown in Figure 4.

The time for switching to conventional setpoint control t_2 is a function of X and dX/dt at t_1, as shown in Figure 5. Here, as in Figure 3, the lines are through the $M = 2$ points only. Knowing t_1, X and dX/dt at t_1, t_2 can be computed using the correlation in Figure 5:

$$t_2 = \left[\frac{1. - X}{b\dfrac{dX}{dt}}\right]_{t_1} + t_1 \qquad (5)$$

where: $b = 0.490 \quad 0.811 \leqq X \leqq 1.$

$b = 0.545 \quad 0.5 \leqq X \leqq 0.811$

Similar curves for $M_1 = 2$, $M_2 = 0$ and $M_1 = 5$, $M_2 = 0$ are shown in Figure 6.

UNIQUENESS

Koppel and Latour (1965) solved the time-optimal control problem analytically to determine the switching curves of second-order systems on the conventional phase plane. Using the same approach the following solution can be obtained:

For the interval $0 \leqq \bar{t} \leqq \bar{t}_1$

$$\frac{d^2X}{dt^2} + \frac{dX}{dt} + AX = AM_1$$

$$\text{at} \quad \bar{t} = 0, \quad X = 0, \quad \frac{dX}{dt} = 0$$

The solution for the system trajectory follows from elementary differential equation theory:

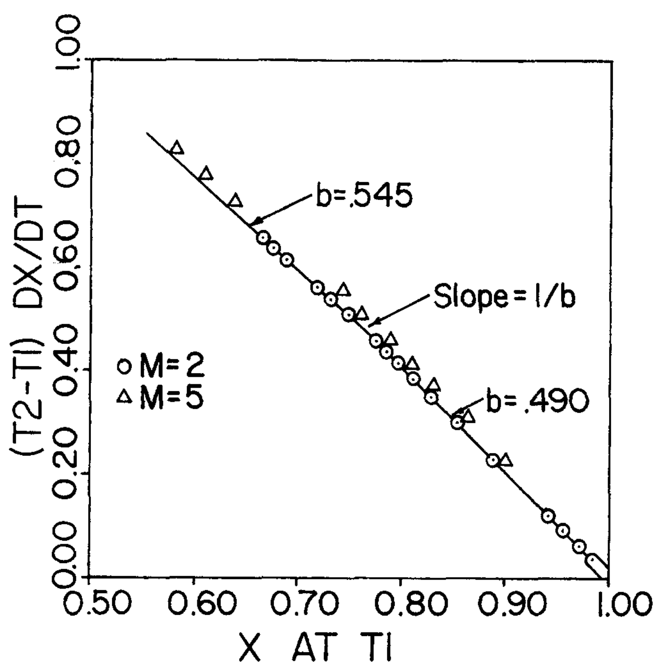

Fig. 5. Correlation for determining t_2. $M_2/M_1 = -1$.

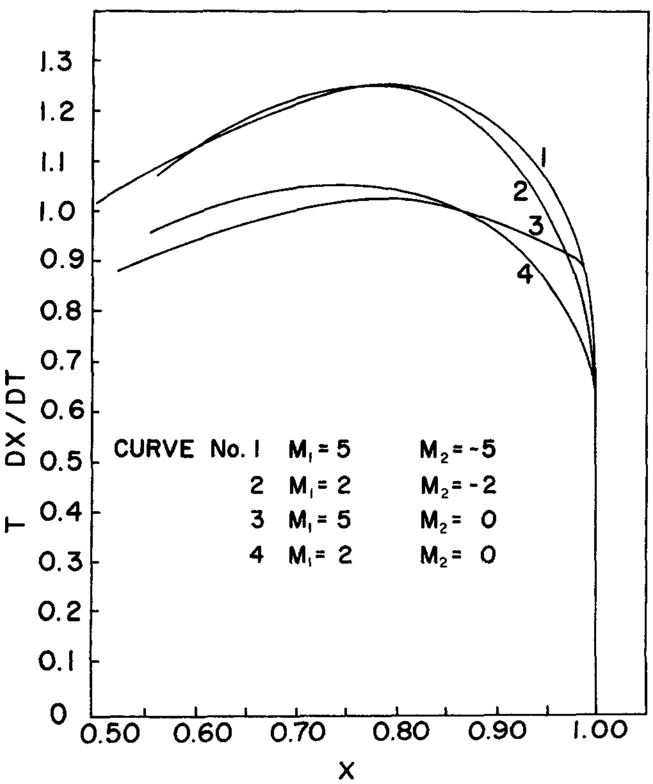

Fig. 4. Switching curves at various values of M_1 and M_2.

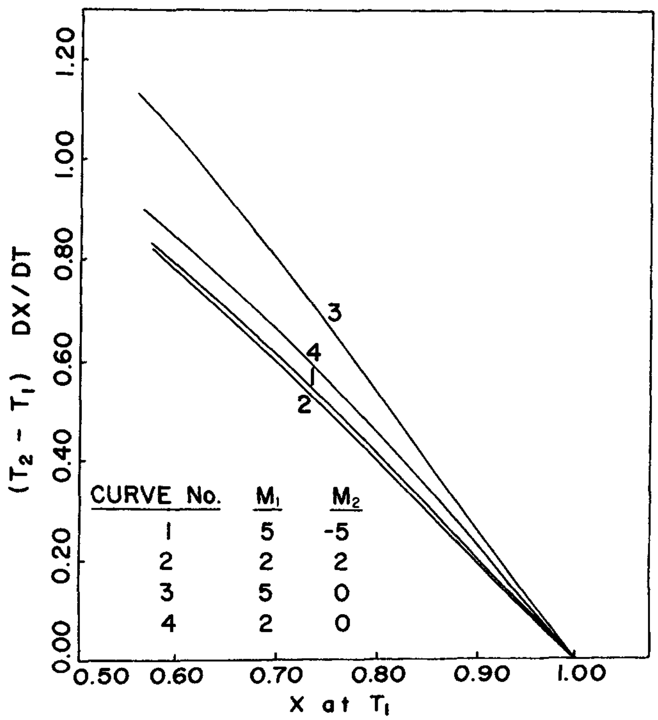

Fig. 6. Correlations for determining t_2 at various values of M_1 and M_2.

$$X = M_1 + \frac{r_2 M_1}{r_1 - r_2} e^{r_1\bar{t}} - \frac{r_1 M_1}{r_1 - r_2} e^{r_2\bar{t}}$$

$$\frac{dX}{d\bar{t}} = \frac{r_1 r_2 M_1}{r_1 - r_2} e^{r_1\bar{t}} - \frac{r_1 r_2 M_1}{r_1 - r_2} e^{r_2\bar{t}}$$

where

$$r_1 = \frac{-1 + \sqrt{1 - 4A}}{2} \quad \text{and} \quad r_2 = \frac{-1 - \sqrt{1 - 4A}}{2}$$

For the interval $\bar{t}_1 \leq \bar{t} \leq \bar{t}_2$

$$\frac{d^2X}{d\bar{t}^2} + \frac{dX}{d\bar{t}} + AX = AM_2$$

$$\text{at} \quad \bar{t} = \bar{t}_2, \quad X = 1, \quad \frac{dX}{dt} = 0$$

The solution is

$$X = M_2 - \frac{r_2(1 - M_2)}{r_1 - r_2} e^{r_1(\bar{t} - \bar{t}_2)} + \frac{r_1(1 - M_2)}{r_1 - r_2} e^{r_2(\bar{t} - \bar{t}_2)}$$

$$\frac{dX}{d\bar{t}} = - \frac{r_1 r_2 (1 - M_2)}{r_1 - r_2} e^{r_1(\bar{t} - \bar{t}_2)} + \frac{r_1 r_2 (1 - M_2)}{r_1 - r_2} e^{r_2(\bar{t} - \bar{t}_2)}$$

The above solutions give the system trajectories in the phase plane before and after $\bar{t}_1$, the time of the first switch. Since the two solutions (before $\bar{t}_1$ and after $\bar{t}_1$) must agree at $\bar{t}_1$:

$$M_1 + \frac{r_2 M_1 e^{r_1\bar{t}_1}}{r_1 - r_2} - \frac{r_1 M_1 e^{r_2\bar{t}_1}}{r_1 - r_2} = M_2 - \frac{r_2(1 - M_2)}{r_1 - r_2} e^{r_1(\bar{t}_1 - \bar{t}_2)} + \frac{r_1(1 - M_2)}{r_1 - r_2} e^{r_2(\bar{t}_1 - \bar{t}_2)}$$

and

$$\frac{M_1 e^{r_1\bar{t}_1}}{r_1 - r_2} - \frac{M_1 e^{r_2\bar{t}_1}}{r_1 - r_2} = - \frac{(1 - M_2)}{r_1 - r_2} e^{r_1(\bar{t}_1 - \bar{t}_2)} + \frac{(1 - M_2)}{r_1 - r_2} e^{r_2(\bar{t}_1 - \bar{t}_2)}$$

These two equations may be solved for the two unknown switching times $\bar{t}_1$ and $\bar{t}_2$ for given values of A, M_1 and M_2

Up to this point the analysis follows Koppel and Latour (1965). Now consider the dimensionless phase plane in which $\bar{t}\, dX/d\bar{t}$ is plotted versus X. For given values of A, M_1, and M_2 the above equations can be solved for $\bar{t}_1$, X at $\bar{t}_1$, $dX/d\bar{t}$ at $\bar{t}_1$, and $\bar{t}\, dX/d\bar{t}$ at $\bar{t}_1$. This gives one point on the switching curve. Now fix M_1 and M_2 and repeat the above calculation for other values of A. This permits the construction of the remainder of the switching curve for given values of M_1 and M_2. Therefore, for fixed M_1 and M_2 there is a unique switching curve in the $\bar{t}\, dX/d\bar{t}$ versus X phase plane. The constant A from the original second-order equation serves as a parameter to locate various points along the switching curve.

Next it should be noted that the $t\, dX/dt$ versus X phase plane (dimensional time) is identical to the $\bar{t}\, dX/d\bar{t}$ versus X phase plane (dimensionless time). This follows because $\bar{t} = B_1 t$, so $\bar{t}\, dX/d\bar{t} = B_1 t\, dX/d(B_1 t) = t\, dX/dt$. Thus the calculated switching curve is valid in the $t\, dX/dt$ versus X phase plane where trajectories can be constructed from observed X and dX/dt data without knowing unsteady state model parameters.

MODELING OF AN OVERDAMPED SECOND-ORDER SYSTEM

The time-optimal and step responses of Equation (3) and the responses of the model obtained by the modeling time-optimal controller algorithm are compared in Figure 7. It should be noted that the parameter values from Equation (3) were not used in determining the switching times for the responses shown in Figure 7. The model parameters of Equations (1) and (2) were determined from the controller during the system time-optimal response shown in Figure 7. The value of t_1 and X at t_1 were measured as 19.86 and 0.945 (see Figure 3) respectively. Figure 8 was entered with X at t_1 and $M = 2$ to obtain the parameter $A = 0.24$. Figure 9 was then entered with $A = 0.24$ to obtain the dimensionless switching time, $\bar{t}_1 = 3.29$. The parameters B_1 and B_2 were then computed.

$$B_1 = \frac{\bar{t}_1}{t_1} = \frac{3.29}{19.86} = 0.166$$

$$B_2 = AB_1^2 = (0.24)(0.166)^2 = 0.0066$$

The model describing Equation (3) is

$$\frac{d^2X}{dt^2} + 0.166\frac{dX}{dt} + 0.0066\,M = 0.0066\,M \tag{6}$$

APPLICATION TO A THIRD-ORDER SYSTEM WITH DEAD TIME

The results described above can be extended to higher-order systems by treating them as second-order systems with dead time θ. When a predicted trajectory $(t - \theta)\, dX/dt + \theta\, dX/dt$, plotted versus $X + \theta\, dX/dt = X + \Delta X$, crosses the switching curve in the dimensionless phase

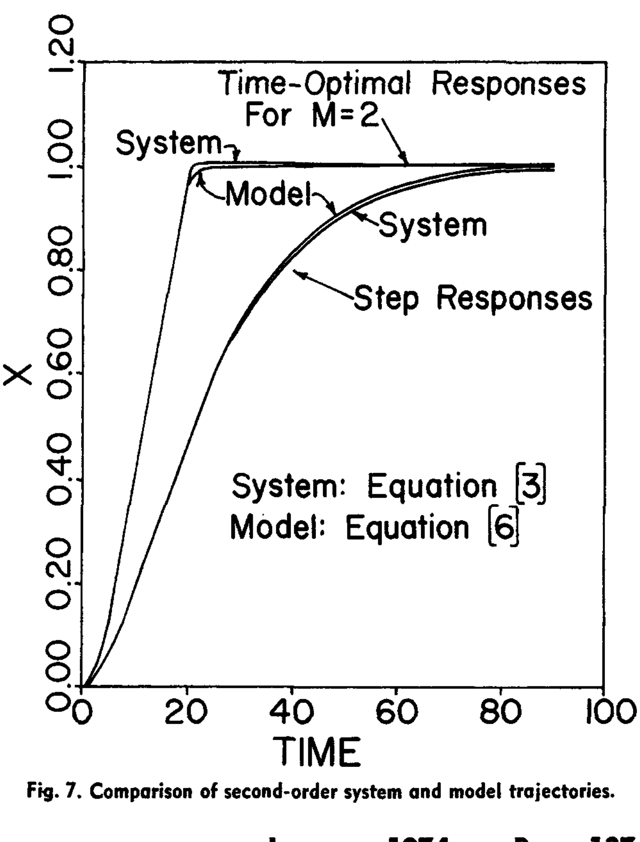

Fig. 7. Comparison of second-order system and model trajectories.

plane, the switch is made between the extremes of the manipulated variable. Figure 11 shows the actual and predicted dimensionless phase plane trajectories for the third-order system:

$$\frac{d^3X}{dt^3} + 0.367\frac{d^2X}{dt^2} + 0.040\frac{dX}{dt} + 0.00133\,X = 0.00133\,M \quad (7)$$

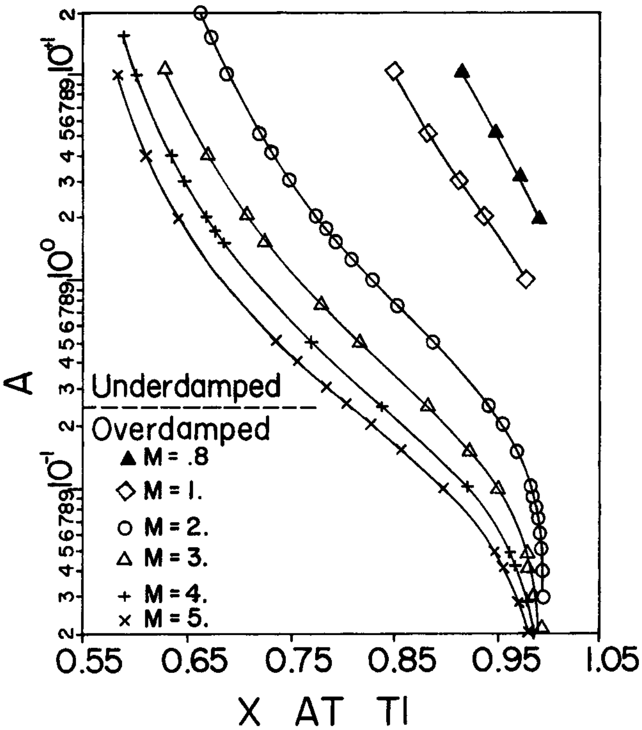

Fig. 8. Percent of setpoint change completed at t_1. $M_2/M_1 = -1$.

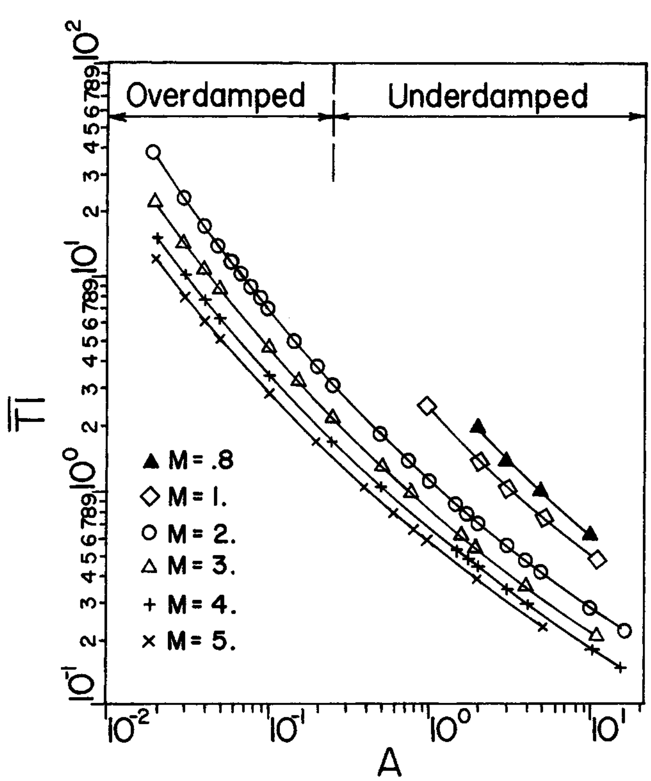

Fig. 9. Dimensionless t_1. $M_2/M_1 = -1$.

with 5 minutes dead time and $M = 2$. The transfer function describing this system is

$$G_{(s)} = \frac{e^{-5s}}{(15s+1)\ (10s+1)\ (5s+1)}$$

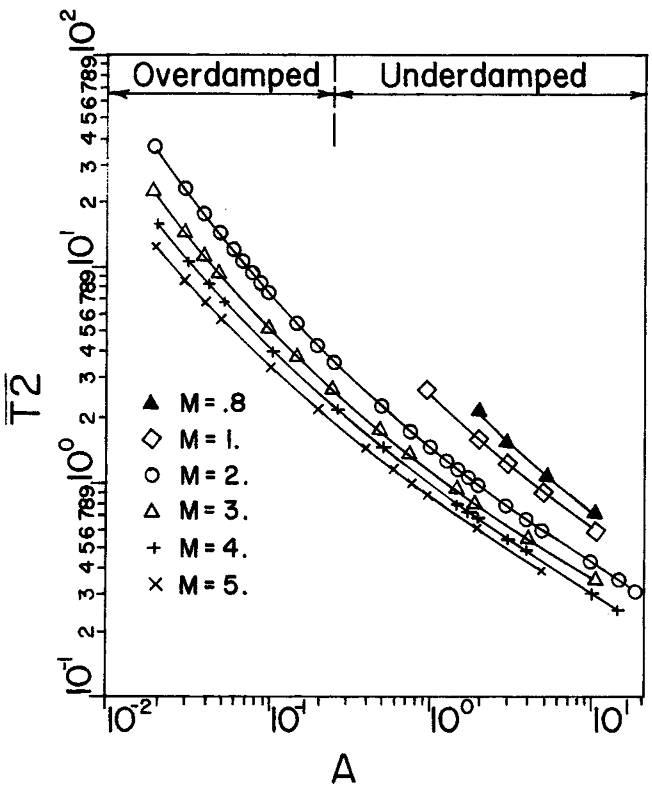

Fig. 10. Dimensionless t_2. $M_2/M_1 = -1$.

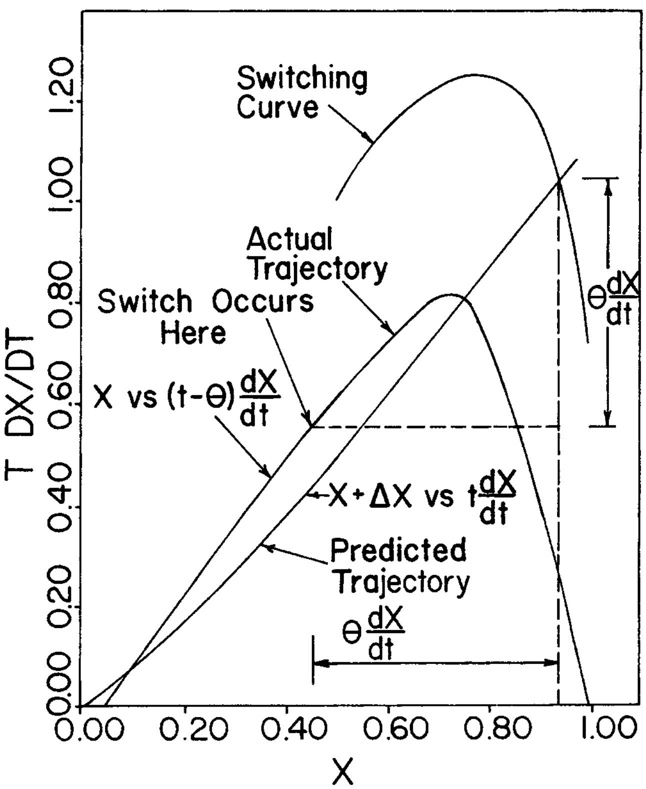

Fig. 11. Dimensionless phase plane for the third-order system described by Equation (7).

It should be noted that in this example t_1 occurs at $X = 0.45$. For a true second-order system, the dimensionless phase plane trajectory is $t\ dX/dt$ versus X. Since this third-order system is being treated as dead time followed by a second-order trajectory, this second-order trajectory is represented in the dimensionless phase plane by $(t - \theta)\ dX/dt$ versus X and is labeled Actual Trajectory. In order to account for the effect of model dead time θ the trajectory must be extended in each direction by an amount $\theta\ dX/dt$ to yield the Predicted Trajectory which actually determines the switching time. The term $\theta\ dX/dt$ corresponds to ΔX on the X axis.

Model parameters can be obtained in the same manner as for a second-order system if $(X + \Delta X)$ at t_1 is used instead of X at t_1. During the time-optimal setpoint change the values of t_1, t_2, and $(X + \Delta X)$ at t_1 were measured as 20.00, 22.26, and 0.942, respectively. The model obtained for this system is

$$\frac{d^2X}{dt^2} + 0.1585\frac{dX}{dt} + 0.0063\,X = 0.0063\,M \qquad (8)$$

with model dead time, $\theta = 10$. The determination of θ is discussed below in the section on tuning. The system and model trajectories are shown in Figure 12.

It should be noted that Equation (8) could be used with Figures 9 and 10 to determine switching times for the system at other values of M.

MODEL QUALITY AND SENSITIVITY

The models obtained using this algorithm closely approximate step changes in the system being modeled. They are as good or better than those obtained using the slope-intercept method (Anderson, 1963) and not quite as good as those obtained by least-squares fit.

The models are not overly sensitive to the measured value of X at t_1. Although an error of one percentage point in the measured value of X at t_1 may cause errors in the model parameters as high as 14%, the error does not greatly affect model trajectories.

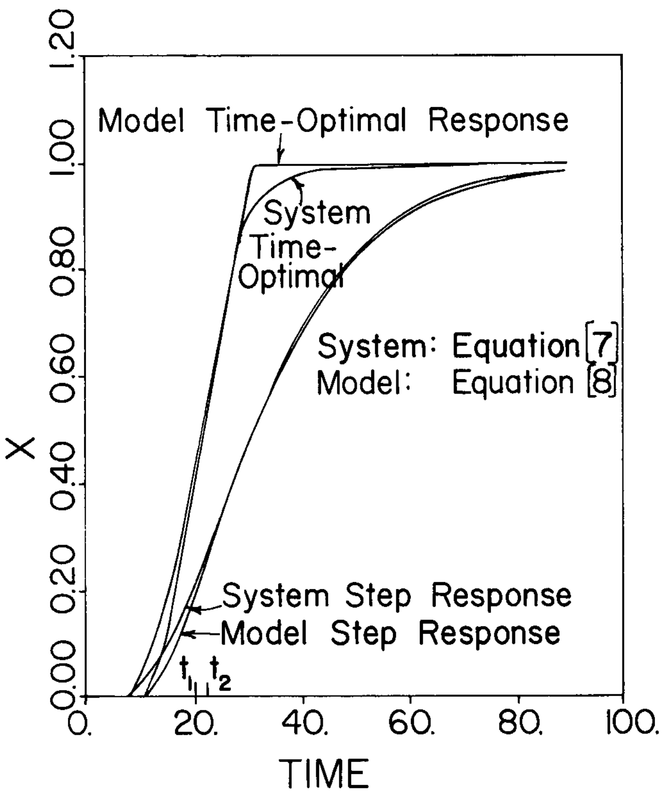

Fig. 12. Comparison of third-order system and second-order model trajectories.

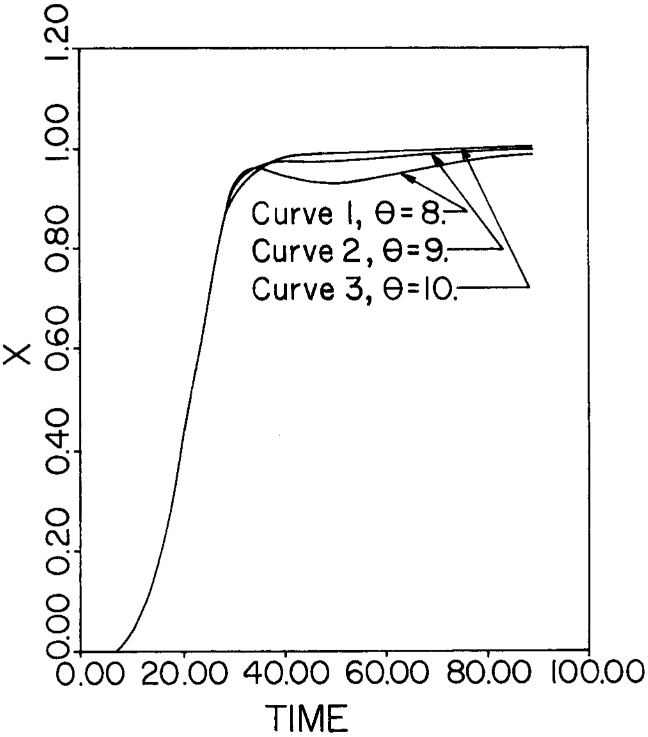

Fig. 13. The tuning of the third-order system described by Equation (7).

TUNING

A single tuning adjustment is required when the controller is put into service; a model dead time θ is determined that gives nearly equal overshoot and undershoot* about the new steady state value $X = 1$. This controller essentially simplifies the parameter identification of a second-order lag plus dead time model from a three-dimensional search to a one-dimensional search. Here as before, it is assumed that we have a steady state model of the system. The controller was tuned in the following manner for the system described by Equation (7). A time-optimal setpoint change was begun with $M = 2$. As the trajectory, shown in Figure 13 as Curve 1, started to rise, an apparent dead time of about $\theta = 8$ was observed and entered into the controller so that it could be used to compute the switching times for Curve 1. A one-dimensional search on θ was then made to determine the value of θ that gave nearly equal overshoot and undershoot about the new steady state value, $X = 1$. A model dead time $\theta = 10.$ produced a satisfactory trajectory which is shown as Curve 3 in Figure 13.

INSENSITIVITY TO CHANGES IN PROCESS DYNAMICS

Once the model dead time θ is determined, the modeling time-optimal controller algorithm is rather insensitive to changes in the system dynamics. This is because the tuning itself is a correction for deviation from second-

* The overshoot and undershoot should be at most only about 1 or 2%. If one should happen to obtain equal overshoot and undershoot of greater value, say 3 or 4%, one might suspect that large nonlinearities exist. In this case, the algorithm can be modified to fit the particular system in a manner similar to that described by Beard (1971).

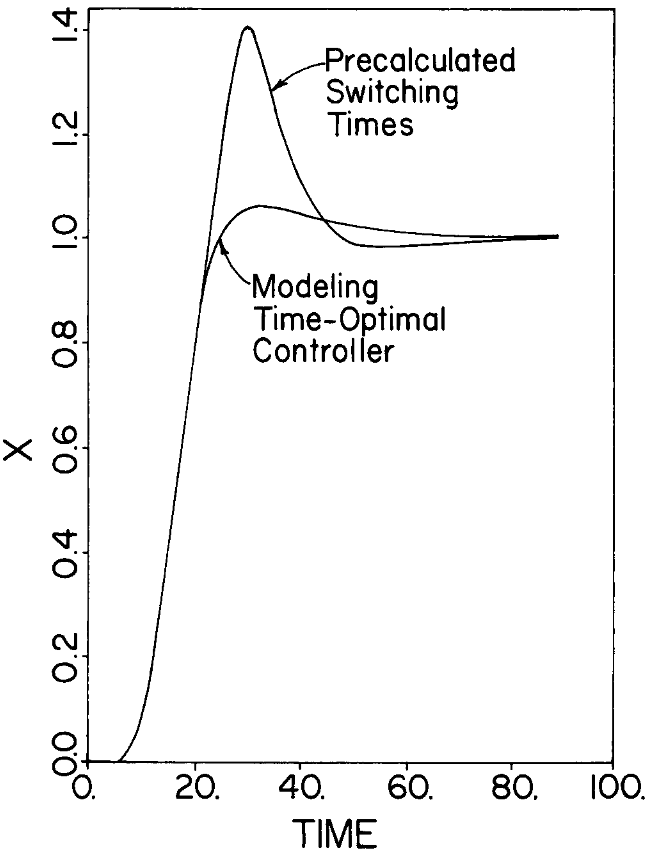

Fig. 14. Time-optimal control of the system described by Equation (9).

order behavior. If the dynamics of the system change, the controller compensates for much of the change and does not require retuning. Control schemes using predetermined models and precalculated switching times cannot compensate for changes in the process dynamics.

The ability of the modeling time-optimal controller algorithm to compensate for undetected changes in process dynamics is demonstrated in Figure 14 where time-optimal trajectories for two different methods are compared. In one method a predetermined least-squares second-order model was obtained for the system described by Equation (7). Using this model, switching times for $M = 2$ were calculated as $t_1 = 22.10$ and $t_2 = 24.90$. In the other method the modeling time-optimal controller algorithm was tuned for the system described by Equation (7). As long as the system was described by Equation (7) either method gave good results. However, the results were different if an undetected change occurred in the process dynamics so that the system was really described by Equation (9):

$$\frac{d^3X}{dt^3} + 0.50\frac{d^2X}{dt^2} + 0.08\frac{dX}{dt} + 0.004\,X = 0.004\,M \tag{9}$$

or

$$G_{(s)} = \frac{e^{-5s}}{(5s+1)\,(10s+1)\,(5s+1)}$$

Figure 14 shows the trajectories that would be obtained for the changed system. The switching times used in Figure 14 are as follows:

	t_1	t_2
Precalculated switching times	22.10	24.90
Modeling time-optimal control	13.16	14.11

ACKNOWLEDGMENTS

Financial support for J. N. Beard, Jr. was provided by an NDEA Fellowship. Computations were performed at the Computer Research Center and Hybrid Computer Facility at Louisiana State University and at the Clemson University Computer Center.

NOTATION

- A = coefficient in dimensionless second-order differential equation
- b = parameter in Figure 4
- B_1, B_2 = coefficients in second-order differential equation
- $f_{(X)}$ = value of switching curve in the dimensionless phase plane
- $G_{(s)}$ = system transfer function
- M, M_1, M_2 = forcing function multipliers
- s = Laplace transform variable
- t = time
- t_1 = time of switch between extremes of the manipulated variable
- t_2 = time of switch to new steady state value of the manipulated variable
- $\bar{t}$ = dimensionless time
- $\bar{t}_1$ = dimensionless t_1
- $\bar{t}_2$ = dimensionless t_2
- X = fraction of setpoint change completed
- ΔX = extrapolation of X in the dimensionless phase plane
- θ = model dead time

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Manuscript received January 17, 1973; revision received September 26 and accepted September 27, 1973.